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ORIGINAL ARTICLE

D-DISTANCE ON GRAPHS

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ABSTRACT

For two vertices u and v of a graph G, the *usual distance* $d(u,v)$, is the length of the shortest path between u and v. Chartand et al introduced the concept of *detour distance* by conserding the length of the longest path between u and v. Kathiresan et al introduced the concept of *superior distance* and *signal distance*.

In some of these distances only the lengths of various paths were considered. By considering the degrees of various vertices present in the path, in addition to the length of the path, in this article we introduced the concept of *D-distance*. We study some properties of this new distance.

1. Introduction

By a graph G, we mean a non-trivial finite undirected connected graph without multiple edges and loops. Following standard notations (for any unexplained notation and terminology we refer [1]) $V(G)$ or V is the vertex set of G and $E(G)$ or E is the edge set of G and $G = G(V, E)$. Let $u, v \in V$ be two vertices of G. The *standard* or *usual distance* $d(u,v)$ between u *and* v is the length of the shortest $u-v$ path in G. Chartrand el al (see [2]) introduceds the concept of *detour distance* in graphs as follows : For two vertices u,v in a graph G, the *detour distance* $D(u,v)$ is defined as the length of the longest $u-v$ path in G. Kathiresan and Marimuthu, in [3], introduced the concept of *superior distance* as follows: For two vertices u,v in a graph G, a $D_{u,v}$ – walk is a $u-v$ walk in G that contains every vertex of $D_{u,v}$ where

$D_{u,v} = N[u] \cup N[v]$. The superior distance is defined as the length of a shortest $D_{u,v}$ −walk. In [4], Kathiresan and Sumathi introduced the concept of *signal distance* in *G*.

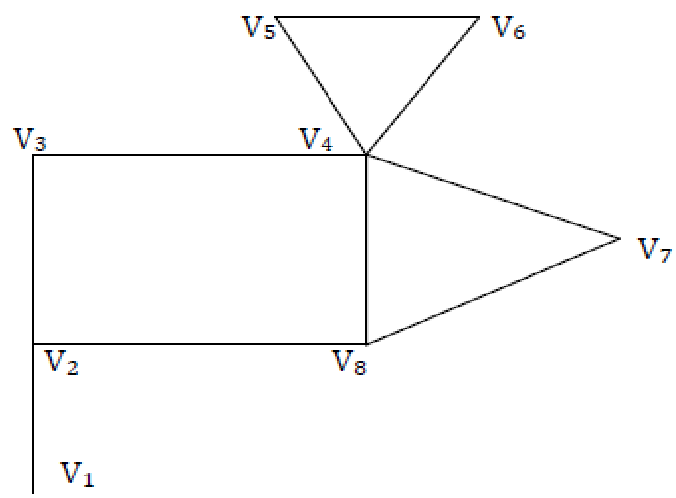
In this article we introduce a new distance, which we call as $D-$*distance* between any two vertices of a graph *G*, and study some of its properties. This distance is significantly different from other distances. In some of the earlier distances, only path length was considered. Here we, in addition, consider the degree of all vertices present in a path while defining its length. Using this length we define the $D-$distance.

2. *D*-distance

Definition 2.1 : If u, v are vertices of a connected graph G, the *D-length* of a $u-v$ path s is defined as $l^D(s) = d(u,v) + \deg(u) + \deg(v) + \sum \deg(w)$ where sum runs over all intermediate vertices w of s.

Definition 2.2 : The $D-distance$ $d^D(u,v)$ between two vertices u, v of a connected graph G is defined as $d^D(u,v) = \min\{l^D(s)\}$ where the minimum is taken over all $u-v$ paths s in G. In otherwords, $d^D(u,v) = \min\{d(u,v) + \deg(u) + \deg(v) + \sum \deg(w)\}$ where the sum runs over all intermediate vertices w in s and minimum is taken over all $u-v$ paths in *G*.

Example 2.3 : In the below graph G



if $u = v_1, v = v_7$, then if S_1 in the path $\{u = v_1, v_2, v_3, v_4, v_7 = v\}$ then *D*-length of the path s_1 is $l^D(S_1) = 4+1+2+3+2+5 = 17$. If S_2 is the path $\{u = v_1, v_2, v_3, v_4, v_8, v_7 = v\}$ then the *D*-length of this path is $l^D(S_2) = 5+1+3+2+5+3+2 = 21$. If S_3 is the path $\{u = v_1, v_2, v_8, v_7 = v\}$ then the *D*-length of this path is $l^D(S_3) = 3+1+3+3+2 = 12$. Similarly, by considering all other paths between u and v we can see that $d^D(u,v)=12$. Further, one can see that the usual distance between u and v is, $d(u,v)=3$.

Remark 2.4 : Observe that for any two vertices u,v of G we have $d(u,v)\le d^D(u,v)$. The equality holds if and only if u,v are identical.

One more

Remark 2.5 : Observe that $D-$distance $d^D(u,v)$ between two vertices u and v of G becomes $\min_s\left\{l(s)+\sum_{w\in V(s)}\deg(w)\right\}$ where the sum runs over all vertices present in s and the minimum is taken over all $u-v$ paths s here $l(s)$ is the length of the path s.

Next we prove an important property about $D-$distance.

Theorem 2.6 : *If G, is any connected graph, then the $D-$distance is a metric on the set of vertices of G.*

Proof : Let G be a connected graph and $u,v\in V(G)$. Then it is clear by definition, that $d^D(u,v)\ge 0$ and $d^D(u,v)=0 \Leftrightarrow u=v$. Also we have $d^D(u,v)=d^D(v,u)$. Thus it remains to show that d^D satisfies the triangle inequality.

Let $u,v,w\in V(G)$. Let P and Q be $u-w$ and $w-v$ paths in G respectively such that $d^D(u,w)=l^D(P)$ and $d^D(w,v)=l^D(Q)$. Let $R=P\cup Q$ be the $u-v$ path obtained by joining P and Q at w. Then

$$\begin{aligned}
d^D(u,w)+d^D(w,v) &= \left(l(P)+\sum_{x\in V(P)}\deg x\right)+\left(l(Q)+\sum_{y\in V(Q)}\deg y\right)\\
&= l(P\cup Q)+\sum_{x\in V(P)}\deg x+\sum_{y\in V(Q)}\deg y\\
&= l(P\cup Q)+\sum_{x\in V(P\cup Q)}\deg x+\deg w\\
&\ge d^D(u,v)+\deg w\\
&\ge d^D(u,v)
\end{aligned}$$

Thus the triangle inequality holds and hence d^D is a metric on the vertex set $V(G)$.

Next we have a consequence of the above proof.

Corollary 2.7 : *For any three vertices* u, v, w *of a graph* G, *we have* $d^D(u,v) \leq$ $d^D(u,w) + d^D(w,v) - \deg w$.

Proposition 2.8 : *In a connected graph* G, *two distinct vertices* u, v *are adjacent if and only if* $d^D(u,v) = \deg u + \deg v + 1$.

Proof : If $u, v \in V(G)$ are adjacent then $d\ (u,v) = 1$ and hence $d^D(u,v) =$ $d(u,v) + \deg u + \deg v = \deg u + \deg v + 1$. Conversely, if $d^D(u,v) = \deg(u) + \deg(v) + 1$, then by definition of $d-$distance we get $d(u,v) + \sum_{w \in V(s)} \deg w = 1$ Hence $d(u,v) = 1$ and $\sum_{w} \deg w = 0$. This implies u and v are adjacent.

3. D-Eccentricity, D-Radius and D-Diameter

This section we begin with some definitions.

Definition 3.1 : The *D-eccentricity* of any vertex $v, e^D(v)$, is defined as the maximum distance from v to any other vertex, i.e., $e^D(v) = \max\{d^D(u,v) : u \in V(G)\}$

Definition 3.2 : Any vertex u for which $d^D(u,v) = e^D(v)$ is called $D-$*eccentric* vertex of v. Further, a vertex u is said to be $D-$*eccentric vertex* of G if it is the $D-$eccentric vertex of some vertex.

Definition 3.3 : The $D-$*radius,* denoted by $r^D(G)$, is the minimum $D-$*eccentricity* among all vertices of G i.e., $r^D(G) = \min\{e^D(v) : v \in V(G)\}$. Similarly the $D-$*diameter,* $d^D(G)$, is the maximum *D-eccentricity* among all vertices of G.

Definition 3.4 : The $D-$*center* of $G, C^D(G)$, is the subgraph induced by the set of all vertices of minimum $D-$*eccentricity.* A graph is called $D-$*self- centered* if $C^D(G) = G$ or equivalently $r^D(G) = d^D(G)$. Similarly, the set of all vertices of maximum $D-$eccentricity is the *periphery* of G.

Remark 3.5 : Observe that since the distance d^D is a metric, we can check easily $r^D(G) \leq d^D(G) \leq 2r^D(G)$. The lower bound is clear from the definition and the upper bound follows from the triangular inequality.

Next we give some properties of $D-$eccentricity.

Theorem 3.6 : *If* u, v *are two adjacent vertices of a connected graph* G, *with* $e^D(u) \geq e^D(v)$, *then* $e^D(u) - e^D(v) \leq \deg(u) + 1$.

Proof: let w be a vertex of G such that $d^D(u,w) = e^D(u)$. Then $e^D(u) = d^D(u,w)$ $\leq d^D(u,v) + d^D(v,w) - \deg(v) \leq d^D(u,w) + e^D(v) - \deg(v)$ (by corollary 2.7). Further since u, v are adjacent, by proposition 2.8, we get $e^D(u) - e^D(v) \leq d^D(u,v) - \deg(v) = \deg(u) + 1$.

Next, we give $D-$radius and $D-$diameter of some families of graphs. To start with, there are graphs for which $D-$radius is same as $D-$diameter .

Proposition 3.7 *: For, complete graphs,* K_n, *on* n *vertices, (* $n \geq 3$ *) we have* $r^D(K_n) = d^D(K_n) = 2n - 1$.

Next, we give examples of families of graphs for which $D-$radius is less the $D-$diameter.

Proposition 3.8 : *For path graphs* P_n *on* n *vertices* $(n \geq 3)$ *we have*

$$r^D(P_n) = \begin{cases} \frac{3n-1}{2} & if\ n\ is\ odd \\ \frac{3n+2}{2} & if\ n\ is\ even \end{cases} \quad \text{and} \quad d^D(P_n) = 3(n-1).$$

Next we give some examples of $D-$self centered graphs.

Proposition 3.9 : *For cycle graphs,* C_n, *with* n *vertices, we have* $r^D(C_{2n}) = d^D(C_{2n}) = 3n + 2$ *and* $r^D(C_{2n+1}) = d^D(C_{2n+1}) = 3n + 2$.

Thus we can see that complete graphs and cycle graphs are $D-$self-centered graphs.

Proposition 3.10 : *For wheel graph,* $W_{1,n}$, *with* $n+1$ *vertices (when* $n \geq 6$) *we have* $r^D(W_{1,n}) = n + 4$, and $d^D(W_{1,n}) = n + 8$

Proposition 3.11 : *For complete biparted graph* $K_{m,n}$, *we have* $r^D(K_{m,n}) = n + 2(m+1)$, and $d^D(K_{m,n}) = m + 2(n+1)$.

Observe that the complete biparted graph $K_{m,m}$ is $D-$self-centered.

Proposition 3.12 : *For star graph,* St_n*, we have* $r^D(St_n) = d^D(St_n) - 2 = n + 2.$

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