A STUDY OF HEAT EQUATION WITH MIXED BOUNDARY CONDITION BY DUAL INTEGRAL EQUATION

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Abstract

In this paper we determine a solution of steady state heat conduction equation inspherical symmetric coordinates subject toa mixed discontinuous boundaryconditions of the third kind acted along a surface of a solid unbounded plate. Herewe use the Hunkle integral transform for solution of the problem. Also weintroduce a Fredholm integral equation of second kind together with dual integralequation. In the classical method we use separation of variables based on theapplication of the dual integral equation method with Bessel function as a kernel.

Key words: Fredholm integral equation of second kind, mixed boundaryConditions

Short Profile

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INTRODUCTION:

Here we present an exact solution of two dimensional heat equations in spherical symmetrical coordinates with discontinuous mixed boundary conditions of second kind on the level surface of semi infinite solid spherical coordinate plates. Inmathematical physics equations the mixed boundary value problems are well known which is given in dual series as [2 ,7]. The solution of the mixed boundaryvalue problem is obtained by separation of variable, Hankle transform which areclassical method and based on the application of dual integral equation method with Bessel function as a kernel. The solution of dual integral equation isintroduced tosome type of Fredholm integral equation of second kind and free term given in the form of improper integral. Such type of integral equations can be solved numerically.Alsothe substitution method, method of discontin uous integrals, regula rization method, infinite series method and other methods given in[10]. For mixed boundary conditions the integ ral transform method is much effective which is used in different areas of the physical and technical application. The main purpose of this paper is to measure the effective change and comparable study to the results given by other integral methods.

Mathematical Formulation of the Problem:

Let us consider a harmonic function = (r,)

Which satisfies the two dimensional steady state heat conduction equation in spherical symmetric co-ordinates with no heat generation for a plane of height h.

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Here the boundary conditionsare

$$(\ )_{r=0} = (\ )_{r\to\infty} = (\ )_{r\to\infty} = 0$$

A linear combination of non homogeneous discontinuous mixed boundary conditions at$\theta = \theta_1$

, $0< r < R$are given by

$$\frac{\partial}{\partial\theta} - Q_1 = -f_1(r) \quad 0< r < R\ ,\theta\ = 0$$

$$\frac{\partial}{\partial\theta} - Q_2 = -f_2(r) \qquad R< r < \infty\ ,\ \theta\ = 0$$

Where $f_1(r)$ and $f_2(r)$ are given functions of r known as heat exchangefunction and R is the radius of the disc. On the surface $\theta = \theta_1$, $r\geq 0$ a homogeneous mixed boundary condition is given by

$$\frac{\partial}{\partial\theta} - Q_3 = 0 \qquad \theta\ = \theta_1, r\geq 0$$

$Q_i = \frac{\alpha_i}{\lambda_i}$ $0 < Q_i < \infty$ Where Q_i , i = 1,2,3 is the coefficient of heat exchange.$\alpha_i$and λ_i are constantsknown as heat exchange coefficients and heat conductivity coefficients respectively (12) . Here the equations (2.1) and (2.3) under the given boundary condition indicates that the boundary surface looses heat by convection and the being heat transfer is according to Newton law of cooling and is proportional to temperature difference to the surrounding temperature. Also at the situation$\theta = \theta_1$ $r\geq 0$ the heat dissipation become by convection from the boundary surface intosurrounding at temperature zero. On the surface level$0< r < R$, $\theta\ = 0$ i.e. inside the disk, the heat exchange function$f_1(r)$ being different from the heat exchange function $f_2(r)$.The heat exchange function$f_2(r)$ is acted outside of the

disc $R < r < \infty$, $\theta\ = 0$. Here R is the line of discontinuity.

The general solution of boundary value problem using (2.1) is

$$(r,) = \int_0^\infty \frac{(n+Q_3)e^{(-2h-\theta)}+(n-Q_3)e^{-n\theta}}{n-Q_3}dn$$

(2.5)

Where $j_o(nr)$ is Bessel function of first kind of order zero [1], n is the separationof variables parameter, $\lambda(n)$ is the unknown function to be determined. Now inabove equation (2.5)using the mixed boundary conditions

$$\int_0^\infty \lambda(n)j_o(nr)K_1(n)\,dn = f_1(r) \quad 0< r < R$$

(2.6)

$$\int_0^\infty \lambda(n)j_o(nr)K_2(n)\,dn = f_2(r) \quad R< r < \infty$$

(2.7)

Where $K_1(n) = \frac{n(n+Q_3)\exp(-2hn)-n(n-Q_3)-Q_1(n+Q_3)\exp(-2hn)-Q_1(n-Q_3)}{n-Q_3}$

$K_2(n) = \frac{n(n+Q_3)\exp(-2hn)-n(n-Q_3)-Q_2(n+Q_3)\exp(-2hn)-Q_2(n-Q_3)}{n-Q_3}$

Now we have to solve the dual integral equation (2.6) and (2.7). For this we put these equations in standard form as

$$\int_0^\infty C(n)j_o(nr)A(n)dn = -f_1(r) \quad 0< r < R$$

(2.8)

$$\int_0^\infty C(n)j_o(nr)\,dn \qquad = -f_2(r) \quad R< r < \infty$$

(2.9)

Where $C(n) = \lambda(n)K_2(n), A(n)=\frac{K_1(n)}{K_2(n)}$, $\lim_{n\to\infty} A(n) = 1$

Now equation (2.7) can be written in following form

$\int_0^\infty C(n)j_o(nr)\,dn \quad = \quad \Psi(r$
$0<r<R - f2(r)\, r<R<\infty$ (2.10)

Where (r) is unknown function defined over the interval (0,R).

Now to determine the function $C(n)$, applying Hunkle inverse Transform to the equation (2.10) we get

$$C(n) = \int_0^R ynj_o(ny)(y)dy - \int_R^\infty ynj_o(ny)f_2(y)dy \qquad (2.11)$$

Now substituting eq. (2.11) into (2.8) and interchanging the order of

integration we get a second kind Fredholm integral equation with unknownfunction (r)as

(r) $+\int_0^R (y)K(r,y)dy=$ F(r)$0< r < R$(2.12)

where the kernel K(r, y) is given by (2.15)

$$K(r,y) = \int_0^\infty ynj_o(ny)j_o(nr)B(n)dn \qquad (2.13)$$

And the free term F(r) is given by

$$F(r) = -f_1(r) + \int_R^\infty \int_0^\infty ynf_2(y)j_o(nr)j_o(ny)B(n)dn\,dy \qquad (2.14)$$

Where B(n) = A(n) −1 and is given by

$$B(n) = \frac{(Q_2 - Q_1)(n-Q_3) + (n+Q_3)\exp(-2hn)}{n(n+Q_3)\exp(-2hn) - n(n-Q_3) - Q_2(n+Q_3)\exp(-2hn) - Q_2(n-Q_3)}$$

The equations (2.13) and (2.14) must satisfy the property [7]

$$\int_0^R |F(r)|dr < \infty ,$$
$$\int_R^\infty \int_0^\infty K^2(n,y)dn\,dy < \infty \qquad (2.15)$$

The above work done is for symmetrical spherical polar coordinates. The similar process should be used for non symmetrical spherical polar coordinatesinvolving Bessel function of first kind$j_v(x)$where $v > -\frac{1}{2}$. Hence the dual integral equation (2.8) and (2.9) takes the form

$$\int_0^\infty C(n)j_v(nr)A(n)dn = -f_1(r) \qquad 0< r < R \qquad (2.16)$$

$$\int_0^\infty C(n)j_v(nr)\,dn = -f_2(r) \qquad R< r < \infty \qquad (2.17)$$

Now using the inverse Hankle transform to the equation (2.16) we get

$$C(n) = \int_0^R ynj_v(ny)(y)dy - \int_R^\infty ynj_v(ny)f_2(y)dy \qquad (2.18)$$

Putting equation (2.18) into the equation (2.16) and then interchanging the order of obtaining equation, we get an integral equation of second

(r) $+ \int_0^R (y)K(r,y)dy$ = F(r)
$0< r < R$(2.19)

Where the kernel K(r, y) is given by

$$K(r,y) = \int_0^\infty ynj_v(ny)j_v(nr)B(n)dn$$

And the free term F(r) is given by

$$F(r) = -f_1(r) + \int_R^{\infty} \int_0^{\infty} y n f_2(y) j_v(nr) j_v(ny) B(n)\, dn\, dy$$

The kernel and free term of the integral equation (2.19) satisfy the condition (2.15).

In the case when v = $\pm 1/2$ the Bessel function equation (2.16) and (2.17) reduced totrigonometric functions and of the dual integral Hankle integral transform replaced by sine or cosine Fourier transforms.

There are several methods on the numerical solution of integral equations (2.12) and (2.19). The single integral equation would be replaced to simultaneous algebraic equations and then matrix techniques are applied. This method becomes more useful with mathematical software packages, Mathematica or matlab.The quadrature method can be also used to solve the integral equations (2.12) and (2.19)since the kernel K(r, y) is continuous and has continuous derivatives with respect to r and y.The quadrature – matrix technique isapplied to the symmetric kernel K(r, y) and the free term F(r) by replacing theintegral to a set of simultaneous algebraic equations.

CONCLUSION:

In this paper a mixed boundary value problem is considered and is solved by integral transform method using Hankle inverse transform and Bessel function. The similar method can be used to solve the problems of mathematical physics using dual integral equation with different areas as heat, electricity,elasticity, electromagnetic etc. The presented method is also applied in efficient manner for solving triple integral equation, quadruple integral equation with Bessel function as a kernel.

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