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FIXED POINT THEOREMS IN FUZZY METRIC SPACES

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ABSTRACT

One of the main problems in the theory of fuzzy topological spaces is to obtain an appropriate and consistent notion of a fuzzy metric space. Many authors have investigated this question and several notions of a fuzzy metric space have been defined and studied.Some common fixed point theorems in complete fuzzy metric spaces are proved which generalize earlier results. We also introduce the concept of R-weak commutatively of type (P) in fuzzy metric spaces. Some related results and illustrative examples are also discussed.In this paper, we state and prove some common fixed point theorems in fuzzy metric spaces. These theorems generalize and improve known results.

KEYWORDS:*Fixed point; Fuzzy metric spaces.*

$$\rho(y_{2n}, y_{2n+1}, t) = \rho(Sx_{2n}, Tx_{2n+1}, t)$$

$$\le q \max \left\{ \begin{matrix} \rho(Ax_{2n}, Bx_{2n+1}, t), \rho(Ax_{2n}, Sx_{2n}, t), \rho(Bx_{2n+1}, Tx_{2n+1}, t), \\ \rho(Ax_{2n}, Tx_{2n+1}, t), \rho(Bx_{2n+1}, Sx_{2n}, t) \end{matrix} \right\}$$

$$= q \max \left\{ \begin{matrix} \rho(y_{2n-1}, y_{2n}, t), \rho(y_{2n-1}, y_{2n}, t), \rho(y_{2n}, y_{2n+1}, t), \\ \rho(y_{2n-1}, y_{2n+1}, t), \rho(y_{2n}, y_{2n}, t) \end{matrix} \right\}$$

$$\le q \max \left\{ \begin{matrix} \rho(y_{2n-1}, y_{2n}, t), \rho(y_{2n}, y_{2n+1}, t), \\ s[\rho(y_{2n-1}, y_{2n}, t) + \rho(y_{2n}, y_{2n+1}, t)] \end{matrix} \right\} \ldots\ldots\ldots\ldots(1)$$

1. INTRODUCTION

One of the most interesting research topics in fuzzy topology is to find an appropriate definition of fuzzy metric space for its possible applications in several areas.It proved a turning point in the development of mathematics when the notion offuzzy set was introduced by Zadeh[2] which laid the foundation of fuzzy mathematics. Consequently the last three decades were very productive for fuzzy mathematicsand the recent literature has observed the fuzzification in almost every direction ofmathematics such as arithmetic, topology, graph theory, probability theory, logicetc. Fuzzy set theory has applications in applied sciences such as neural networktheory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing,control theory, communication etc. No wonder that fuzzy fixed point theory hasbecome an area of interest for specialists in fixed point theory, or fuzzy mathematicshas offered new possibilities for fixed point theorists. In 1965, the theory of fuzzy sets was investigated by Zadeh[2]. In 1981, Heilpern[3]first introduced the concept of fuzzy contractive mappings and proved a fixed point theorem for these mappings in metric linear spaces. His result is a generalization of the fixed point theorem for point-to-set maps of Nadler[4]. Therefore, several fixed point theorems for types of fuzzy contractive mappings have appeared (see, for instance[1, 5, 7, 8&9].In this paper, we state and prove some common fixed point theorems in fuzzy metric spaces. These theorems generalize and improve known results.

There are various ways to define a fuzzy metric space, here we adopt the notion that, the distance between objects is fuzzy, and the objects themselves may be fuzzy or not.

2. BASIC PRELIMINARIES

The definitions and terminologies for further discussions are taken from Heilpern[3]. Let (X, d) be a metric linear space. A **fuzzy set** in X is a function with domain X and values in [0, 1]. If A is a fuzzy set and $x \in X$, then the function-value $A(x)$ is called the **grade of membership** of x in A.

The collection of all fuzzy sets in X is denoted by I(X).
Let A∈I(X) and $\alpha \in [0, 1]$. The ***α*-level set** of A, denoted by $A\alpha$, is defined by

$$A_\alpha = \{x : A(x) \geqslant \alpha\} \quad \text{if} \quad \alpha \in (0, 1], \quad A_0 = \overline{\{x : A(x) > 0\}},$$

whenever $\overline{B}$ is the closure of set (nonfuzzy) B.

Definition 2.1.
A fuzzy set A in X is an **approximate quantity** iff its α-level set is a nonempty compact convex subset (nonfuzzy) of X for each $\alpha \in [0, 1]$ and $sup_{x\in X}A(x) = 1$.
The set of all approximate quantities, denoted by $W(X)$, is a subcollection of I(X).

Definition 2.2.
Let $A, B \in W(X)$, $\alpha \in [0, 1]$ and $CP(X)$ be the set of all nonempty compact subsets of X. Then

$$p_\alpha(A, B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x, y), \quad \delta_\alpha(A, B) = \sup_{x\in A_\alpha, y\in B_\alpha} d(x, y) \quad \text{and}$$

$$D_\alpha(A, B) = H(A_\alpha, B_\alpha),$$

where H is the **Hausdorff metric** between two sets in the collection $CP(X)$. We define the following functions

$$p(A, B) = \sup_\alpha p_\alpha(A, B), \quad \delta(A, B) = \sup_\alpha \delta_\alpha(A, B) \quad \text{and}$$

$$D(A, B) = \sup_\alpha D_\alpha(A, B).$$

It is noted that $p\alpha$ is non-decreasing function of α.

Definition 2.3.
Let $A, B \in W(X)$. Then A is said to be **more accurate** than B (or B includes A), denoted by $A \subset B$, if $A(x)$ ⍰ $B(x)$ for each $x \in X$.
The relation ⊂ induces a partial order on $W(X)$.

Definition 2.4.
Let X be an arbitrary set and Y be a metric linear space. F is said to be a **fuzzy mapping** if F is a mapping from the set X into $W(Y)$, i.e., $F(x) \in W(Y)$ for each $x \in X$.
The following proposition is used in the sequel.

Proposition 2.1.
If $A, B \in CP(X)$ and $a \in A$, then there exists $b \in B$ such that $d(a, b)$ ⍰ $H(A, B)$.
Following Beg and Ahmed[10], let (X, d) be a metric space. We consider a sub-collection of I(X) *denoted by $W^*(X)$. Each fuzzy set $A \in W^*(x)$, its α-level set is a nonempty compact subset (nonfuzzy) of X for each $\alpha \in [0, 1]$. It is obvious that each element $A \in W(X)$ leads to $A \in W^*(X)$ but the converse is not true.*
The authors introduced the improvements of the lemmas in Heilpern[3] as follows.

Lemma 2.1.
If $\{x0\} \subset A$ for each $A \in W^*(X)$ and $x0 \in X$, then $p\alpha(x0, B)$ ⍰ $D\alpha(A, B)$ for each $B \in W^*(X)$.

Lemma 2.2.
$p\alpha(x, A)$ ⍰ $d(x, y) + p\alpha(y, A)$ for all $x, y \in X$ and $A \in W^*(X)$.

Lemma 2.3.

Let $x \in X$, $A \in W^*(X)$ and $\{x\}$ be a fuzzy set with membership function equal to a characteristic function of the set $\{x\}$. Then $\{x\} \subset A$ if and only if $p\alpha(x, A) = 0$ for each $\alpha \in [0, 1]$.

Lemma 2.4.

Let (X, d) be a complete metric space, $F: X \to W^*(X)$ be a fuzzy map and $x0 \in X$. Then there exists $x1 \in X$ such that $\{x1\} \subset F(x0)$.

Remark 2.1.

It is clear that Lemma 2.4 is a generalization of corresponding lemma in Arora and Sharma[1] and Proposition 3.2 in Lee and Cho[7].

Let Ψ be the family of real lower semi-continuous functions F: $[0, \infty)^6 \to R$, $R :=$ the set of all real numbers, satisfying the following conditions:

(ψ_1)

F is non-increasing in 3rd, 4th, 5th, 6th coordinate variable,

(ψ_2)

there exists $h \in (0, 1)$ such that for every $u, v \geq 0$ with

(ψ_{21})

$F(u, v, v, u, u + v, 0) \leq 0$ or (ψ_{22}) $F(u, v, u, v, 0, u + v) \leq 0$, we have $u \leq h v$, and

(ψ_3)

$F(u, u, 0, 0, u, u) > 0$ for all $u > 0$.

3. MAIN RESULTS

In 2000, Arora and Sharma[1] proved the following result.

Theorem 3.1.

Let (X, d) be a complete metric space and $T1$, $T2$be fuzzy mappings from X into $W(X)$. If there is a constant q, $0 \leq q < 1$, such that, for each $x, y \in X$,

$D(T_1(x),T_2(y)) \leq q\max\{d(x,y),p(x,T_1(x)),p(y,T_2(y)),p(x,T_2(y)),p(y,T_1(x)\}$,

then there exists $z \in X$ such that $\{z\} \subset T1(z)$ and $\{z\} \subset T2(z)$.

Remark 3.1.

If there is a constant q, $0 \leq q < 1$, such that, for each $x, y \in X$,equation(1)

$D(T_1(x),T_2(y)) \leq q\max\{d(x,y),p(x,T_1(x)),p(y,T_2(y))\}$,

then the conclusion of Theorem 3.1 remains valid. This result is considered as a special case of Theorem 3.1.Beg and Ahmed[10] generalized Theorem 3.1 as follows.

Theorem 3.2.

*Let (X, d) be a complete metric space and $T1$, $T2$be fuzzy mappings from X into $W^*(X)$. If there is a $F \in \Psi$ such that, for all $x, y \in X$,*equation(2)

$F(D(T_1(x),T_2(y)),d(x,y),p(x,T_1(x)),p(y,T_2(y)),p(x,T_2(y)),p(y,T_1(x))) \leq 0$,

then there exists $z \in X$ such that $\{z\} \subset T_1(z)$ and $\{z\} \subset T_2(z)$.

Widely inspired by a paper of Tas et al.[11], we give another different generalization of Theorem 3.1 with contractive condition (1) as follows.

Theorem 3.3.

*Let (X, d) be a complete metric space and $T1$, $T2$be fuzzy mappings from X into $W^*(X)$. Assume that there exist $c1, c2, c3 \in [0, \infty)$ with $c1 + 2c2 < 1$ and $c2 + c3 < 1$ such that, for all $x, y \in X$,*equation (3)

$D^2(T_1(x),T_2(y)) \leq c_1\max\{d^2(x,y),p^2(x,T_1(x)),p^2(y,T_2(y))\}+c_2\max\{p(x,T_1(x)p(x,T_2(y)),p(y,T_1(x))p(y,T_2(y))\}+c_3p(x,T_2(y))p(y,T_1(x))$.

Then there exists $z \in X$ such that $\{z\} \subset T1(z)$ and $\{z\} \subset T2(z)$.

Proof.

Let x_0 be an arbitrary point in X. Then by Lemma 2.4, there exists an element $x_1 \in X$ such that $\{x_1\} \subset T_1(x_0)$. For $x_1 \in X$, $(T_2(x_1))_1$ is nonempty compact subset of X. Since $(T_1(x_0))_1$, $(T_2(x_1))_1 \in CP(X)$ and $x_1 \in (T_1(x_0))_1$, then Proposition 2.1 asserts that there exists $x_2 \in (T_2(x_1))_1$ such that $d(x_1,x_2) \leq D_1(T_1(x_0), T_2(x_1))$. So, we obtain from the inequality $D(A, B) \leq D\alpha(A, B)$ $\forall \alpha \in [0, 1]$ that

$$\begin{aligned} d^2(x_1,x_2) &\leqslant D_1^2(T_1(x_0),T_2(x_1)) \leqslant D^2(T_1(x_0),T_2(x_1)) \\ &\leqslant c_1 \max\{d^2(x_0,x_1), p^2(x_0,T_1(x_0)), p^2(x_1,T_2(x_1))\} \\ &+c_2 \max\{p(x_0,T_1(x_0))p(x_0,T_2(x_1)), \\ &\quad p(x_1,T_1(x_0))p(x_1,T_2(x_1))\} \\ &+c_3 p(x_0,T_2(x_1))p(x_1,T_1(x_0)) \\ &\leqslant c_1 \max\{d^2(x_0,x_1), d^2(x_1,x_2)\} \\ &+c_2 d(x_0,x_1)[d(x_0,x_1)+d(x_1,x_2)]. \end{aligned}$$

If $d(x_1, x_2) > d(x_0, x_1)$, then we have
$d^2(x_1,x_2) \leq (c_1+2c_2)d^2(x_1,x_2)$,

which is a contradiction. Thus,

$d(x_1,x_2) \leq h d(x_0,x_1)$,
where $h = \sqrt{c_1 + 2c_2} < 1$. Similarly, one can deduce that

$d(x_2,x_3) \leq h d(x_1,x_2)$.

By induction, we have a sequence (x_n) of points in X such that, for all $n \in N \cup \{0\}$,

$$\{x_{2n+1}\} \subset T_1(x_{2n}), \quad \{x_{2n+2}\} \subset T_2(x_{2n+1}).$$

It follows by induction that $d(x_n, x_{n+1}) \leq h^n(x_0, x_1)$. Since

$$\begin{aligned} d(x_n,x_m) &\leqslant d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \ldots + d(x_{m-1},x_m) \\ &\leqslant h^n d(x_0,x_1) + h^{n+1} d(x_0,x_1) + \ldots + h^{m-1} d(x_0,x_1) \\ &\leqslant \frac{h^n}{1-h} d(x_0,x_1), \end{aligned}$$

then $\lim_{n,m\to\infty} d(x_n, x_m) = 0$. Therefore, (x_n) is a Cauchy sequence. Since X is complete, then there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$. Next, we show that $\{z\} \subset T_i(z)$, $i = 1, 2$. Now, we get from Lemma 2.1 and Lemma 2.2 that
$p_\alpha(z,T_2(z)) \leq d(z,x_{2n+1})+p_\alpha(x_{2n+1},T_2(z)) \leq d(z,x_{2n+1})+D_\alpha(T_1(x_{2n}),T_2(z))$,
for each $\alpha \in [0, 1]$. Taking supremum on α in the last inequality, we obtain thatequation(4)
$p(z,T_2(z)) \leq d(z,x_{2n+1})+D(T_1(x_{2n}),T_2(z))$.
From the inequality (3), we have thatequation-(5)
$D^2(T_1(x_{2n}),T_2(z)) \leq c_1\max\{d^2(x_{2n},z),p^2(x_{2n},T_1(x_{2n})),p^2(z,T_2(z))\}+c_2\max\{p(x_{2n},T_1(x_{2n}))p(x_{2n},T_2(z)),p(z,T_1(x_{2n}))p(z,T_2(z))\}+c_3p(x_{2n},T_2(z))p(z,T_1(x_{2n})) \leq c_1\max\{d^2(x_{2n},z),d^2(x_{2n},x_{2n+1}),p^2(z,T_2(z))\}+c_2\max\{d(x_{2n},x_{2n+1})p(x_{2n},T_2(z)),d(z,x_{2n+1})p(z,T_2(z))\}+c_3p(x_{2n},T_2(z))d(z,x_{2n+1})$.
Letting $n \to \infty$ in the inequalities (4) and (5), it follows that
$p(z,T_2(z)) \leqslant \sqrt{c_1}p(z,T_2(z))$.

Since $\sqrt{c_1} < 1$, we see that $p(z, T_2(z)) = 0$. So, we get from Lemma 2.3 that $\{z\} \subset T_2(z)$. Similarly, one can be shown that $\{z\} \subset T_1(z)$.

Remark 3.2.

(I) Condition (3) is not deducible from condition (2) since the function F from $[0, \infty)^6$ into $[0, \infty)$ defined as $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^2 - c_1 \max\{t_2^2, t_3^2, t_4^2\} - c_2 \max\{t_3 t_5, t_6 t_4\} - c_3 t_5 t_6$, for all $t_1, t_2, t_3, t_4, t_5, t_6 \in [0, \infty)$, where $c_1, c_2, c_3 \in [0, \infty)$ with $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1$, does not general satisfy condition (ψ_3). Indeed, we have that $F(u,u,0,0,u,u)=u^2-c_1u^2-c_3u^2$, for all $u > 0$ and does not imply that $F(u, u, 0, 0, u, u) > 0$ for all $u > 0$. It suffices to consider $c_1 = \frac{3}{4}$, $c_2 = \frac{1}{9}$, $c_3 = \frac{1}{2}$ and then $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1$ but $F(u, u, 0, 0, u, u) < 0$ for all $u > 0$. Therefore, Theorem 3.2 and Theorem 3.3 are two different generalizations of Theorem 3.1 with contractive condition (1).

(II) If there exist $c_1, c_2, c_3 \in [0, \infty)$ with $c_1 + 2c_2 < 1$ and $c_2 + c_3 < 1$ such that, for all $x, y \in X$, $\delta^2(T_1(x),T_2(y)) ⍰ c_1\max\{d^2(x,y),p^2(x,T_1(x)),p^2(y,T_2(y))\}+c_2\max\{p(x,T_1(x))p(x,T_2(y)),p(y,T_1(x))p(y,T_2(y))\}+c_3p(x,T_2(y))p(y,T_1(x)$, then the conclusion of Theorem 3.3 remains valid. This result is considered as a special case of Theorem 3.3 because $D(F_1(x), F_2(y))$ ⍰ $\delta(F_1(x), F_2(y))$ [12, page 414]. Moreover, this result generalizes Theorem 3.3 of Park and Jeong[8].

Example 3.1.

Let $X = [0, 1]$ endowed with the metric d defined by $d(x, y) = |x - y|$. It is clear that (X, d) is a complete metric space. Let $T_1 = T_2 = T$. Define a fuzzy mapping T on X such that for all $x \in X$, $T(x)$ is the characteristic function for $\{\frac{3}{4}x\}$. For each $x, y \in X$,

$$\begin{aligned} D^2(T(x), T(y)) &= \frac{9}{16} d^2(x,y) \\ &\leqslant c_1 \max\{d^2(x,y), p^2(x,T(x)), p^2(y,T(y))\} + c_2 \\ &\quad \times \max\{p(x,T(x))p(x,T(y)), p(y,T(x))p(y,T(y))\} \\ &\quad + c_3 p(x,T(y))p(y,T(x)), \end{aligned}$$

where $c_1 = \frac{9}{16} < 1$ and $c_2 = c_3 = 0$. The characteristic function for $\{0\}$ is the fixed point of T.

The following theorem generalizes Theorem 3.3 to a sequence of fuzzy contractive mappings.

Theorem 3.4.

Let (Tn: $n \in N \cup \{0\}$) be a sequence of fuzzy mappings from a complete metric space (X, d) into $W^*(X)$. Assume that there exist $c1, c2, c3 \in [0, \infty)$ with $c1 + 2c2 < 1$ and $c2 + c3 < 1$ such that, for all $x, y \in X$,

$$\begin{aligned} D^2(T_0(x), T_n(y)) \leqslant c_1 \max\{&d^2(x,y), p^2(x,T_0(x)), p^2(y,T_n(y))\} \\ &+ c_2 \max\{p(x,T_0(x))p(x,T_n(y)), \\ &p(y,T_0(x))p(y,T_n(y))\} \\ &+ c_3 p(x,T_n(y))p(y,T_0(x)) \quad \forall n \in N. \end{aligned}$$

Then there exists a common fixed point of the family (Tn: $n \in N \cup \{0\}$).

Proof.

Putting $T_1 = T_0$ and $T_2 = Tn$ $\forall n \in N$ in Theorem 3.3. Then, there exists a common fixed point of the family (Tn: $n \in N \cup \{0\}$).

Remark 3.3.

If there is a □ ∈ Φ such that, for all $x, y \in X$,

$$D^2(T_0(x), T_n(y)) \leqslant c_1 \max\{d^2(x, y), p^2(x, T_0(x)), p^2(y, T_n(y))\}$$
$$+ c_2 \max\{p(x, T_0(x))p(x, T_n(y)),$$
$$p(y, T_0(x))p(y, T_n(y))\}$$
$$+ c_3 p(x, T_n(y))p(y, T_0(x)) \quad \forall n \in N.$$

then the conclusion of Theorem 3.4 remains valid. This result is considered as a special case of Theorem 3.4 for the same reason in Remark 3.2(I).

CONCLUSION

We defined the notion of fuzzy cone metric space which is a generalization of fuzzy metric spaces and then the topology induced by this space. By using these definitions we gave some topological properties, such as Hausdorfness, first countability. The cone version of fuzzy Banach contraction theorem is also stated here. So one can study, by using these results, on the other fix point theorems, similar topological properties of this space and problems related to convergence of a sequence.

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