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GRT

THE MINIMUM GLOBAL DOMINATING ENERGY OF A GRAPH

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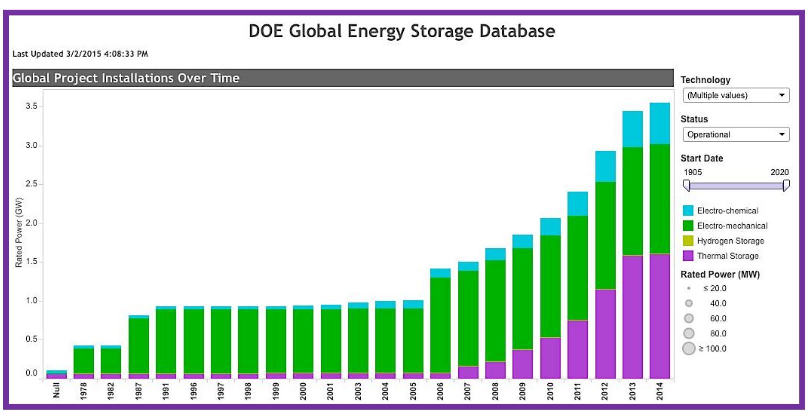
Abstract

In a graph $G(V,E)$, a subset $D \subseteq V$ is called a global dominating set of G, if D is a dominating set of both G and $\bar{G}$. The global domination number $\gamma_g(G)$ is the minimum cardinality of a minimal global dominating set in G. In this paper, we study the minimum dominating energy, denoted by $E_{GD}(G)$, of a graph G and minimum double dominating energy of G. We compute the minimum global dominating energies of complete graph, complete bipartite graph, star graph and cocktail party graph. Upper and lower bounds for $E_{GD}(G)$ are established.

Mathematics Subject Classification: 05C50, 05C69.

Key words: global dominating set, global domination number, global dominating matrix, global dominating eigenvalues, minimum global dominating energy, Minimum Double Dominating Energy.



1. INTRODUCTION

In this paper, by a graph $G(V,E)$ we mean a simple graph that is finite, have no loops no multiple and directed edges. We denoted by n and m to the number of its vertices and edges, respectively. We refer the reader to Harary book [9] for more graph theoretical analogist not defined here. A subset $D \subseteq V(G)$ is called a dominating set of G if every vertex $v \in V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)\ of\ G$ is the minimum cardinality of a minimal dominating set in G. A subset $D \subseteq V(G)$ is called a global dominating set in G if D is a dominating set of both G and $\bar{G}$. The global domination number $\lambda_g(G)$ is the minimum cardinality of a minimal global dominating set in G. Any global dominating set in a graph with minimum cardinality is called a minimum global dominating set. The concept of global domination in graph was introduced by Sampathkumar [18], while Kulli and Janakiram [14] have introduced the concept of total global dominating sets. Also, there are more studies and details in this concept can be see it in [4, 20, 21, 19] and the references cited there in. While for more details in domination theory of graphs we refer to Haynes et al. book [10].

The concept of energy of a graph was introduced by I. Gutman [7] in the year 1978. Let G be a graph with n vertices and m edges and let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy $E(G)\ of\ G$ is defined to be the sum of the absolute values of the eigenvalues of G, i.e.

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

For more details on the mathematical aspects of the theory of graph energy see [2, 8, 15]. The basic properties including various upper and lower bounds for energy of a graph have been established in [16, 17], and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [5, 6].

Recently C. Adiga et al [1] defined the minimum covering energy, $E_C(G)$ of a graph which depends on its particular minimum cover C. Further, minimum dominating energy, Laplacian minimum dominating energy and minimum dominating distance energy of a graph G can be found in [11, 12, 13] and the references cited there in.

Motivated by these papers, we study the minimum global dominating energy $E_{GD}(G)$ of a graph G and minimum double dominating Energy. We compute minimum global dominating energies of some standard graphs. Upper and lower bounds for E_{GD}(G) are established. It is possible that the upper dominating energy that we are considering in this paper may be have some applications in chemistry as well as in other areas.

2. THE MINIMUM GLOBAL DOMINATING ENERGY OF A GRAPH

Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, \ldots\ldots, v_n\}$ and edge set $E(G)$. Let D be a global dominating set in G. The global domination number $\gamma_g(G)$ of G is the cardinality of a smallest global dominating set in G. Any global dominating set D with cardinality equals to $\gamma_g(G)$ is called a minimum global dominating set of G. The minimum global dominating matrix of G is then $n \times n$ matrix, denoted by $A_{GD}(G) = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1, & if\ v_i v_j \in E; \\ 1, & if\ \ i = j\ \ and\ \ v_i \in D; \\ 0, & otherwise. \end{cases}$$

Characteristic polynomial of $A_{GD}(G)$ is denoted by

$f_n(G, \lambda) = \det(\lambda I - A_{GD}(G))$.

The minimum global dominating eigen values of a graph G are the eigenvalues of $A_{GD}(G)$ .Since $A_{GD}(G)$is real and symmetric, its eigen values are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \ldots\ldots \geq \lambda_n$. The minimum global dominating energy of G is defined as:

$$E_{GD}(G) = \sum_{I=1}^{n} |\lambda_i|.$$

We first compute the minimum global dominating energy of a graph in Figure 1

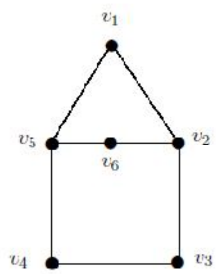


Figure-1

Let G be a graph in Figure 1, with vertices set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Then G has more one set as a minimum global dominating set. For example, $D1 = \{v_1, v_3, v_4\}$. Then

$$A_{GD_1}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $A_{GD_1}(G)$ is
$f_n(G,\lambda) = \lambda^6 - 3\lambda^5 - 5\lambda^4 + 12\lambda^3 + 9\lambda^2 - 9\lambda - 5.$

Hence, the minimum global dominating eigenvalues are $\lambda_1 \approx 3.2263$, $\lambda_2 \approx 1.9102$, $\lambda_3 \approx 1.0000$, $\lambda_4 \approx -0.4939$, $\lambda_5 \approx -1$, $\lambda_6 \approx -1.6426$.
Therefore the minimum global dominating energy of G is $E_{GD_1}(G) \approx 9.2730$.
But if we take another minimum global dominating set of G, namely $D_2 = \{v_2, v_3, v_6\}$,
we get that

$$A_{GD_2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The characteristic polynomial of $A_{GD_2}(G)$ is
$f_n(G,\lambda) = \lambda^6 - 3\lambda^5 - 5\lambda^4 + 11\lambda^3 + 8\lambda^2 - 8\lambda - 4.$

The minimum global dominating eigenvalues are $\lambda_1 \approx 3.3795$, $\lambda_2 \approx 1.6708$, $\lambda_3 \approx 1$, $\lambda_4 \approx$

-0.4399, $\lambda_5 \approx -1$, $\lambda_6 \approx -1.6105$. Therefore the minimum global dominating energy of G is $E_{GD_2}(G) \approx 9.1007$.

This example illustrates the fact that the minimum global dominating energy of a graph G depends on the choice of the minimum global dominating set. i.e. the minimum global dominating energy is not a graph invariant.

In the following section, we introduce some properties of characteristic polynomial of minimum global dominating matrix of a graph G.

Theorem 2.1. Let G be a graph of order n, size m, minimum global dominating set D and let

$$f_n(G,\lambda) = c_0\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots \ldots\ldots .. + c_n.$$

be the characteristic polynomial of the minimum global dominating matrix of a graph G.

Then

1 $c_0 = 1$.

2 $c_1 = -|D|$

3. $c_2 = \binom{|D|}{2} - 2$.

Proof. 1. From the definition of $f_n(G,\lambda)$.

2. Since the sum of diagonal elements of $A_{GD}(G)$ is equal to $|D|$. The sum of determinants of all 1×1 principal submatrices of $A_{CD}(G)$ is the trace of $A_{GD}(G)$,which evidently is equal to $|D|$. Thus, $(-1)^1 c_1 = |D|$.

3. $(-1)^2 c_2$ is equal to the sum of determinants of all 2×2 principal submatrices of $A_{GD}(G)$, that is

$$c_2 = \sum_{1\le i<j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$

$$= \sum_{1\le i<j\le n} (a_{ii}a_{jj} - a_{ij}a_{ji})$$

$$= \sum_{1\le i<j\le n} a_{ii}a_{jj} - \sum_{1\le i<j\le n} {a_{ij}}^2$$

$$= \binom{|D|}{2} - m$$

Theorem 2.2. Let G be a graph of order n. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A_{GD}(G)$. Then

(i) $\sum_i^n \lambda_i = |D|$.

$(ii)\ \sum_i^n {\lambda_i}^2 = |D| + 2m.$

Proof. (i) Since the sum of the eigen values of $A_{GD}(G)$ is the trace of $A_{GD}(G)$, then

$$\sum_i^n \lambda_i = \sum_{i=1}^n a_{ii} = |D|.$$

(ii) Similarly the sum of squares eigenvalues of $A_{GD}(G)$ is the trace of $(A_{GD}(G))^2$.

Then,

$$\sum_{i=1}^n {\lambda^2}_i = \sum_{i=1}^n \sum_{j=1}^n a_{ij}a_{ji}$$

$$= \sum_{i=1}^n {a^2}_{ii} + \sum_{i\neq j}^n a_{ij}a_{ji}$$

$$= \sum_{i=1}^n {a^2}_{ii} + 2\sum_{i\neq j}^n {a^2}_{ij}$$

$$= |D| + 2m$$

Bapat and S.Pati [3], proved that if the graph energy is a rational number then it is an even integer. Similar result for minimum global dominating energy is given in the following theorem.

Theorem 2.3. Let G be a graph with a minimum global dominating set D. If the minimum global dominating energy $E_{GD}(G)$ of G is a rational number, then

$$E_{GD}(G) \equiv |D|(mod 2).$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the minimum global dominating eigen values of a graph G of which $\lambda_1, \lambda_2, \ldots, \lambda_r$ are positive and the rest are non-positive, then

$$\sum_{i=1}^n |\lambda_i| = (\lambda_1 + \lambda_2 + \cdots \ldots . + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + \cdots \ldots .. + \lambda_n)$$

$$= 2(\lambda_1 + \lambda_2 + \cdots \ldots . + \lambda_r) - (\lambda_1 + \lambda_2 + \cdots \ldots . + \lambda_r).$$

$$= 2q - |D|. where\ q = \lambda_1 + \lambda_2 + \cdots \ldots . + \lambda_r.$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_r$ are algebraic integers, so is their sum. Hence $(\lambda_1 + \lambda_2 + \ldots + \lambda_r)$ must be an integer if $E_{GD}(G)$ is rational.

Theorem 2.4. For $n \geq 1$, the minimum global dominating energy of the complete graph $K_n, is\ E_{GD}(K_n) = n.$

Proof. For the complete graphs K_n, the minimum global dominating set is all vertices, i.e. $D = V(K_n)$.

Then,

$$A_{GD}(K_n) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n\times n}$$

$$= \lambda^{n-1}(\lambda - n)$$

The characteristic polynomial of $A_{GD}(K_n)$ is

$$f_n(K_n, \lambda) = \begin{vmatrix} \lambda-1 & -1 & \cdots & -1 \\ -1 & \lambda-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda-1 \end{vmatrix}_{n\times n}$$

The spectrum of K_n is

$$GD\ Spec(K_n) = \begin{pmatrix} 0 & n \\ n-1 & 1 \end{pmatrix}$$

Therefore, the minimum global dominating energy of the complete graph is

$$E_{GD}(K_n) = n.$$

∎

Theorem 2.5. *For the complete bipartite graph $K_{r,r}$, for $r \geq 2$, the minimum global dominating energy is equal to*

$$(r+1) + \sqrt{r^2 + 2r - 3}$$

Proof. For the complete bipartite graph $K_{r,r}$ with vertex set $V = (V_1, V_2)$ where V_1 and V_2 are the partite sets of its, $V_1 = \{v_1, v_2, \cdots, v_r\}$ and $V_2 = \{u_1, u_2, \cdots, u_r\}$.

The minimum global dominating set is $D = \{v_1, u_1\}$. Then

$$A_{GD}(K_{r,r}) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2r\times 2r}$$

The characteristic polynomial of $A_{GD}(K_{r,r})$ is

$$f_n(K_{r,r},\lambda)=\begin{vmatrix} \lambda-1 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ 0 & \lambda & \cdots & 0 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & \lambda-1 & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2r\times 2r}$$

$$= \lambda^{2r-4}[\lambda^2 + (r-1)\lambda - (r-1)][\lambda^2 - (r+1)\lambda + (r-1)]$$

The spectrum of $K_{r,r}$ is $GDSpec(K_{r,r})$

$$\begin{pmatrix} 0 & \frac{(1-r)-\sqrt{r^2+2r-3}}{2} & \frac{(1-r)+\sqrt{r^2+2r-3}}{2} & \frac{(1+r)-\sqrt{r^2-2r+5}}{2} & \frac{(1-r)-\sqrt{r^2-2r+5}}{2} \\ 2r-4 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Therefore, the minimum global dominating energy of the complete bipartite graph is $E_{GD}(K_{r,r}) = (1+r) + \sqrt{r^2+2r-3}$

Theorem 2.6. For $n \geq 2$,the minimum global dominating energy of a star graph $K_{1,n-1}$ is less than or equal to $2 + 2\sqrt{n-2}$

Proof. Let $K_{1,n-1}$ be a star graph with vertex set $V = \{v_0, v_1, v_2, \cdots, v_{n-1}\}$, where $v_0$is the central vertex. Since , $\overline{K_{n-1}} = K_1 \cup K_{N-1}$ it follows that the minimum global dominating set of $K_{1,n-1}$ is $D = \{v_0, v_i\}$ for any $i = 1,2,\ldots,n-1$.

Hence, if we choose $D = \{v_0, v_i\}$,then the minimum global dominating matrix of $K_{1,n-1}$ is

$$A_{GD}(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}_{n\times n}$$

The characteristic polynomial of $A_{GD}(K_{1,n-1})$ is

$$f_n(K_{1,n-1},\lambda) = \begin{vmatrix} \lambda-1 & -1 & -1 & \cdots & -1 \\ -1 & \lambda-1 & 0 & \cdots & 0 \\ -1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{n\times n}$$

$$= (\lambda)^{n-3}[\lambda^3 - 2\lambda^2 - (n-2)\lambda + (n-2)]$$

$$\leq (\lambda)^{n-3}[\lambda^3 - 2\lambda^2 - (n-2)\lambda + 2(n-2)]$$

The spectrum of $K_{1,n-1}$is

$$GD\ Spec(K_{1,n-1}) = \begin{pmatrix} 0 & 2 & \sqrt{n-2} & -\sqrt{n-2} \\ n-3 & 1 & 1 & 1 \end{pmatrix}$$

Therefore, the minimum global dominating energy of a star graph is

$$E_{GD}(K_{1,n-1}) \leq 2 + 2\sqrt{n-2}$$

Definition 2.7. *The cocktail graph, denoted by $K_{2\times p}$, is a graph having vertex set $V(K_{2\times p}) \cup_{i=1}^{p}\{u_i, v_i\}$ and edge set $E(K_{2\times p}) = \{u_iu_j, v_iv_j, u_iv_j, v_iu_j : 1 \leq i < j \leq p\}$, i.e,$n = 2p, m = \frac{p^2-3p}{2}$ and for ever $v \in V(K_{2\times p}), d(v) = 2p-2$.*

Theorem 2.8. For the cocktail party graph $K_{2\times p}$ of order $2p$, for $p \geq 3$, the minimum global dominating energy is equal to $(2p-1) + \sqrt{5}(p-1)$

Proof. For cocktail party graphs $K_{2\times p}$ the minimum domination set is D$= \{u_i | 1 \leq i \leq p\}$.Hence, for cocktail party graphs the minimum global dominating matrix is

$$A_{GD}(K_{2\times p}) = \begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \end{pmatrix}_{2p\times 2p}$$

The characteristic polynomial of $AGD(K_{2\times p})$ is

$$f_n(K_{2\times p},\lambda)=\begin{vmatrix} 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & 1 & 1 \\ 1 & 1 & 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \end{vmatrix}_{2p\times 2p}$$

$$= [\lambda^2-(2p-1)\lambda+(p-1)][\lambda^2+\lambda-1]^{(p-1)}$$

The spectrum of $K_{2\times p}$ is

$$GD\ Spec(K_{2\times p})=\begin{pmatrix} \frac{(2p-1)-\sqrt{4p^2-8p+2}}{2} & \frac{(2p-1)+\sqrt{4p^2-8p+2}}{2} & \frac{-1-\sqrt{5}}{2} & \frac{-1+\sqrt{5}}{2} \\ 1 & 1 & p-1 & p-1 \end{pmatrix}$$

Therefore, the minimum global dominating energy

$$E_{GD}(K_{2\times p}) = (2p-1)+\sqrt{5}(p-1).$$

Theorem 2.9. Let G be a graph of order n and size m. Then

$$\sqrt{2m+\gamma_g(G)} \le E_{GD}(G) \le \sqrt{n(2m+\gamma_g(G)}$$

Proof. Consider the Couchy-Schwartiz inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

By choose $a_i = 1 and\ b_i = |\lambda_i|$, we get

$$(E_{GD}(G))^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 \le \left(\sum_{i=1}^{n}1\right)\left(\sum_{i=1}^{n}\lambda^2{}_i\right)$$
$$\le n(2m+|D|)$$
$$\le n\big(2m+\gamma_g(G)\big).$$

Therefore, the upper bound is holds. For the lower bound, since

$$\left(\sum_{i=1}^{n}|\lambda_i|\right)^2 \ge \sum_{i=1}^{n}\lambda_i{}^2.$$

Then, $E_{GD}(G))^2 \ge \sum_{i=1}^{n}\lambda^2{}_i = 2m+|D| = 2m+\gamma_g(G)$

Therefore, $E_{GD}(G) \ge \sqrt{2m+\gamma_g(G)}.$

Similar to McClellands [17] bounds for energy of a graph, bounds for $E_{GD}(G)$ are given in the following theorem.

Theorem 2.10. Let G be a graph of order n and size m, respectively.

If $P = det(A_{GD}(G))$, then

$$E_{GD}(G) \geq \sqrt{2m + \gamma_g(G) + n(n-1)P^{\frac{2}{n}}}$$

$$(E_{GD}(G))^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)\left(\sum_{i=1}^{n} |\lambda_i|\right) = \sum_{i=1}^{n} |\lambda_i|^2 + 2\sum_{i \neq j} |\lambda_i||\lambda_j|.$$

$$\frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i||\lambda_j| \geq \left(\prod_{i \neq j} |\lambda_i||\lambda_j|\right)^{1/[n(n-1)]}.$$

$$\begin{aligned}
(E_{GD}(G))^2 &\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i \neq j} |\lambda_i||\lambda_j|\right)^{1/[n(n-1)]} \\
&\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left(\prod_{i=j} |\lambda_i|^{2(n-1)}\right)^{1/[n(n-1)]} \\
&= \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)\left|\prod_{i \neq j} \lambda_i\right|^{2/n} \\
&= 2m + \gamma_g(G) + n(n-1)P^{2/n}.
\end{aligned}$$

3.MINIMUM DOUBLE DOMINATING ENERGY:

Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, v_3, \ldots \ldots .., v_n\}$ and edge set E. A subset $D' \subseteq V$ is a double dominating set if D' is a dominating set and every vertex of $V - D'$ is adjacent to atleast two vertices in D'. The double Domination number $\gamma_{x2}(G)$ is the minimum cardinality taken over all the minimal double dominating sets of G.

Let D' be the minimum double dominating set of a graph.The minimum double dominating matrix of G is the $n \times n$ matrix defined by $A_D(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & if\ v_i v_j \in E \\ 1 \ \ if\ i = j & and\ v_i \in D' \\ 0 & otherwise \end{cases}$$

The characteristic polynomial of $A_D(G)$ is denoted by $f_n(G, \lambda) = det\,(\lambda I - A_D(G))$.

The minimum double dominating eigen values of the graph G are the eigen values of $A_D(G)$.

Since $A_D(G)$ is real and symmetric, its eigen values are real numbers and are labeled in non-increasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq, \ldots ., \geq \lambda_n$.the minimum double dominating energy of G is defined as $E_D(G) = \sum_{i=1}^{n} |\lambda_i|$.

Example 3.1. Let G be a cycle C_4 on 4 vertices u_1, u_2, u_3, u_4 with minimum double dominating set $D' = \{u_1, u_3\}$. Then

$$A_D(C_4) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The Characteristic polynomial of $A_D(C_4)$ is $\lambda^4 - 2\lambda^3 - 3\lambda^2 + 4\lambda$, the minimum double dominating eigen values are 0, 1, $\frac{1+\sqrt{7}}{2}$, $\frac{1-\sqrt{7}}{2}$, and the minimum double dominating energy is $E_D(C_4) = 1 + \sqrt{17}$.

Theorem 3.2. *Let G be a graph with n vertices and m edges.*

If $\lambda_1, \lambda_2, \lambda_3, , \ldots, \lambda_n$ are the eigen values of $A_D(G)$, then $\sum_{i=1}^{n} {\lambda_1}^2 = 2|E| + |D'|$.

Proof: The sum of the squares of the eigen values of $A_D(G)$ is the trace of $A_D(G)^2$.

$$\sum_{i=1}^{n} {\lambda_1}^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} a_{ji}$$

$$= 2\sum_{i<j} (a_{ij})^2 + \sum_{i=1}^{n} (a_{ij})^2$$

$$= 2|E| + |D'|$$

$$= 2m + |D'|.$$

Theorem 3.3. *Let G be a simple graph with n vertices, m edges and let D' be a double dominating set of G and $F = |det\, A_D(G)|$ then ,*

$$\sqrt{2m + |D'| + n(n-1)f^{\frac{2}{n}}} \leq E_D(G) \leq \sqrt{n(2m + |D'|)}$$

Proof: Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq, \ldots, \geq \lambda_n$ be the eigen values of $A_D(G)$.

By Cauchy-Schwarz inequality, $\sum_{i=1}^{n} (a_i b_i)^2 \leq \sum_{i=1}^{n} ({a_i}^2) (\sum_{i=1}^{n} ({b_i}^2))$.

Let $a_i = 1$, $b_i = |\lambda_i|$,

$$E_D(G)^2 = (\sum_{i=1}^{n} |\lambda_i|)^2 \leq n\left(\sum_{i=1}^{n} |\lambda_i|^2\right) = n\sum_{i=1}^{n} {\lambda_i}^2 = n(2m + |D'|$$

$$E_D(G) \leq \sqrt{n(2m + |D'|}$$

$$[E_D(G)]^2 = (\sum_{i=}^{n} |\lambda_i|)^2 = \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i=j} |\lambda_i| |\lambda_j|$$

From the inequality between the arithmetic and geometric mean,

We obtain $\frac{1}{\mathrm{n(n-1)}}\sum_{i=j}|\lambda_i|\,|\lambda_j| \geq \left(\prod_{i=j}|\lambda_i||\lambda_j|\right)^{\frac{1}{\mathrm{n(n-1)}}}$

$$\sum_{i\neq j}|\lambda_i|\,|\lambda_j| \geq n(n-1)\left[\prod_{i=1}^{n}|\lambda_i|^{2(n-1)}\right]^{\frac{1}{\mathrm{n(n-1)}}}$$

$$\geq n(n-1)\left[\prod_{i=1}^{n}|\lambda_i|\right]^{\frac{2}{\mathrm{n}}}$$

$$\geq n(n-1)\left|\prod_{i=1}^{n}|\lambda_i|\right|^{\frac{2}{\mathrm{n}}}$$

$$\geq n(n-1)|det A_D(G)\,|^{\frac{2}{\mathrm{n}}}$$

$$\sum_{i\neq j}|\lambda_i|\,|\lambda_j| \geq n(n-1)F^{\frac{2}{n}}$$

$$[E_D(G)]^2 \geq \sum_{i=1}^{n}|\lambda_i|^2 + n(n-1)F^{\frac{2}{n}}$$

$$\geq (2\mathrm{m} + |D'| + n(n-1)F^{\frac{2}{n}}$$

$$[E_D(G)] \geq \sqrt{2\mathrm{m} + |D'| + n(n-1)F^{\frac{2}{n}}}$$

Definition 3.4. The crown graph $S_n{}^o$ for an integer $n \geq 2$ is the graph with vertex set $v = \{u_1, u_2, \dots\dots, u_n, v_1, v_2, v_{3,} \dots\dots\dots, v_n\}$ and the edge set $\{u_i v_j : 1 \leq i, j \leq n, i \neq j\}$.

Theorem 3.5. For any $n \geq 2$, the *double dominating energy of the crown graph* $S_n{}^o$ *is equal to* $2 + 2(n-3) + \sqrt{n^2 - 2n + 9} + \sqrt{n^2 + 2n - 7}$.

Proof: The crown graph $S_n{}^o$ with vertex set $v = \{u_1, u_2, \dots\dots, u_n, v_1, v_2, v_{3,} \dots\dots\dots, v_n\}$ the minimum double dominating set $D' = \{u_1, u_2, v_1, v_2\}$.

$$\text{Then } A_{D'}(S_n^o) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 & 1 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & \cdots & 1 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ . & . & . & . & . & . & . & . & . & . \\ 1 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Characteristic polynomial is

$$\begin{bmatrix} \lambda-1 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \lambda-1 & 0 & \cdots & 0 & -1 & 0 & -1 & \cdots & -1 \\ 0 & 0 & \lambda & \cdots & 0 & -1 & -1 & 0 & \cdots & -1 \\ . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & \cdots & \lambda & -1 & -1 & -1 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \lambda-1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & -1 & \cdots & -1 & 0 & \lambda-1 & 0 & \cdots & 0 \\ -1 & -1 & 0 & \cdots & -1 & 0 & 0 & \lambda & \cdots & 0 \\ . & . & . & . & . & . & . & . & . & . \\ -1 & -1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

Characteristic equation is

$$\lambda(\lambda-2)(\lambda+1)^{n-3}(\lambda-1)^{n-3}\big(\lambda^2+(n-3)\big)\lambda-(2n-4)(\lambda^2-(n-1)\lambda-2)=0$$

Minimum double dominating eigen values are

$$\lambda = 0\,, \lambda = 2\,, \lambda = -1\,, (n-3\; times), \lambda = 1(n-3\; times)$$

$$\lambda = \frac{(n-1)\pm\sqrt{n^2-2n+9}}{2}\ (one\ time\ each)$$

$$\lambda = \frac{(3-n)\pm\sqrt{n^2+2n-7}}{2}\ (one\ time\ each)$$

Minimum double dominating energy

$$E_{D'}({S_n}^o) = 2-2(n-3)+\sqrt{n^2-2n+9}+\sqrt{n^2+2n-7}.$$

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